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Int. J. Heat Mass Transfer. Vol. 21, pp. 1264–1266

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STABILITY OF A LAYER OF FLUID SUBJECTED TO CONVECTIVE BOUNDARY CONDITIONS

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NOMENCLATURE

a, $= \lambda L$, dimensionless wave number;
g, acceleration of gravity;
h_1, h_2, heat-transfer coefficients;
H_j, $= h_j L/k$, Biot number, $j = 1, 2$;
k, thermal conductivity of fluid;
L, thickness of the fluid layer;
Pr, Prandtl number;
T, temperature;
$\bar{T}_{w1}, \bar{T}_{w2}$, mean temperatures of the lower and upper surface, respectively;
$T_{\infty 1}, T_{\infty 2}$, temperatures f the outside environments;
t, time;
w, velocity component in the z-direction;
z, coordinate normal to the walls.

Greek symbols

α, ν, thermal diffusivity and kinematic viscosity;
β^*, dimensionless rest-state temperature gradient;
η, $= z/L$, dimensionless coordinat ;
δ, angle measured from horizontal;
γ, coefficient of thermal expansion.

Superscripts

$'$, $\hat{}$, refer to disturbance quantities.

FOR AN infinite horizontal layer of fluid confined between two isothermal plates with lower plate hotter than the upper one, the transition takes place at a critical Rayleigh number 1708 [1–6]. There are numerous other factors that affect the initiation of convective flow patterns in the fluid [7–15]. In all of these and other investigations the boundaries confining the fluid are assumed at prescribed temperatures. Only in [16, 17] the problem of stability is considered for a horizontal layer only with one of the surfaces subjected to convective boundary condition and by neglecting the heat capacity of the walls. The purpose of this study is to investigate the stability of fluid confined between two inclined parallel layers subjected to general convective boundary conditions at both surfaces.

ANALYSIS

Consider a layer of fluid between two parallel plates subjected to a negative temperature gradient in the direction perpendicular to the plates (i.e. z-direction). Fluid is incompressible, Newtonian, the physical properties are constant except for the density which appears in the body force (i.e. Boussinesq approximation) and that the wall admittance is negligible. We restrict our analysis to the range of inclinations such that the stability in the conduction regime with longitudinal disturbances will be dominant (i.e. $\delta \simeq 75°$ for $Pe \simeq 0.7$) [6, 15].

The disturbance equations for the type of stability problem considered here are well known [15, 18]; they are written as

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w' = \gamma g \nabla_{xy}^2 T' \cos\delta \qquad (1a)$$

$$\left(\frac{\partial}{\partial t} - \alpha\nabla^2\right)T' = -\frac{d\bar{T}}{dz}w'. \qquad (2a)$$

Subject to the boundary conditions

$$w' = \frac{\partial w'}{\partial z} = 0 \text{ at } z = 0 \text{ and } z = L. \qquad (1b)$$

$$\begin{aligned} -k\frac{\partial T'}{\partial z} + h_1 T' &= 0 \text{ at } z = 0 \\ k\frac{\partial T'}{\partial z} + h_2 T' &= 0 \text{ at } z = L \end{aligned} \qquad (2b)$$

where for horizontal layer

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla_{xy}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

while $\partial/\partial x = 0$ for inclined layer;
and w' and T' are the disturbance quantities for velocity normal to the walls and the temperature $d\bar{T}/dz$ is the constant negative temperature gradient. The disturbance quantities w' and T' are written as

$$\begin{aligned} F'(x, y, z, t) &= F(z)\exp[i(k_1 x + k_2 y) + pt], \\ F &\equiv w \text{ or } T, \end{aligned} \qquad (3)$$

where

$$(k_1^2 + k_2^2)^{1/2} \equiv \lambda = \text{the wave number}$$

and p, in general, is a complex number, and $k_1 = 0$ for inclined layer.

Then the solution (3) is introduced into the system of equations (1) and (2); the resulting expressions in the dimensionless form are given as

$$[P - (D^2 - a^2)](D^2 - a^2)w^*(\eta) = -a^2\theta^*(\eta) \qquad (4a)$$

$$[PPr - (D^2 - a^2)]\theta^*(\eta) = Raw^*(\eta)\cos\delta \qquad (5a)$$

and the boundary conditions as

$$w^* = 0,\quad \frac{dw^*}{d\eta} = 0 \quad \text{at} \quad \eta = 0 \text{ and } \eta = 1 \tag{4b}$$

$$-\frac{d\theta^*}{d\eta} + H_1\theta^* = 0 \quad \text{at} \quad \eta = 0 \tag{5b}$$

$$\frac{d\theta^*}{d\eta} + H_2\theta^* = 0 \quad \text{at} \quad \eta = 1 \tag{5c}$$

where various dimensionless quantities are defined as

$$\left.\begin{aligned} &a^2 = L^2\lambda^2,\quad D^2 = \frac{d^2}{d\eta^2},\quad H_i = \frac{h_i L}{k},\quad i = 1 \text{ or } 2,\quad P = \frac{pL^2}{\nu},\\ &Ra = \frac{\beta^* g\gamma(T_{\infty 1} - T_{\infty 2})L^3}{\alpha\nu} = \frac{g\gamma(\bar{T}_{w1} - \bar{T}_{w2})L^3}{\alpha\nu},\\ &w^* = \frac{\hat{w}/\cos\delta}{L^2\gamma g(T_{\infty 1} - T_{\infty 2})/\nu},\quad \theta^* = \frac{\hat{T}}{T_{\infty 1} - T_{\infty 2}},\quad \eta = \frac{z}{L},\quad \beta^* \equiv \frac{1}{\dfrac{1}{H_1} + \dfrac{1}{H_2} + 1}. \end{aligned}\right\} \tag{6}$$

To show that the principle of the exchange of stabilities is valid for the considered problem, i.e. $\delta m(P) = 0$, we obtained the following expression

$$\frac{Ra\cos\delta}{a^2} = \frac{\theta_0 + \displaystyle\int_0^1 [D\theta^* D\tilde{\theta}^* + (a^2 + \tilde{P}Pr)\theta^*\tilde{\theta}^*]\,d\eta}{\displaystyle\int_0^1 [D^2w^* D^2\tilde{w}^* + (2a^2 + P)Dw^* D\tilde{w}^* + a^2(a^2 + P)w^*\tilde{w}^*]\,d\eta} \tag{7}$$

where $\tilde{}$ denotes complex conjugate and

$$\theta_0 \equiv [H_2\theta^*(1)\tilde{\theta}^*(1) + H_1\theta^*(0)\tilde{\theta}^*(0)].$$

This result implies that for $Ra\cos\delta > 0$, the imaginary part of P is zero. Then we set $P = 0$ in equations (4a) and (5a) to characterize the marginal state of instability. To solve the stability problem we represent $\theta^*(\eta)$ in a series of orthogonal functions that satisfy the boundary conditions (5b) and (5c) as [19].

$$\theta^*(\eta) = \sum_{m=1}^{\infty} A_m\theta_m(\eta) \tag{8a}$$

where

$$\theta_m(\eta) = \beta_m\cos\beta_m\eta + H_1\sin\beta_m\eta. \tag{8b}$$

β_m's are the positive roots of

$$\tan\beta = \frac{\beta(H_1 + H_2)}{\beta^2 - H_1H_2} \tag{8c}$$

and the functions $\theta_m(\eta)$ are orthogonal as

$$\int_0^1 \theta_m(\eta)\theta_n(\eta)\,d\eta = \begin{cases} 0 & \text{if} \quad m \neq n \\ N_m & \text{if} \quad m = n \end{cases} \tag{8d}$$

where the norm N_m is given as

$$N_m = \frac{1}{2}\left[(\beta_m^2 + H_1^2)\left(1 + \frac{H_2}{\beta_m^2 + H_2^2}\right) + H_1\right]. \tag{8e}$$

The above solution for $\theta^*(\eta)$ is introduced into equation (4a) for $P = 0$, together with the expression for $w^*(\eta)$ chosen as

$$w^*(\eta) = a^2\sum_{m=1}^{\infty} A_m w_m(\eta). \tag{9}$$

One finds

$$(D^2 - a^2)^2 w_m(\eta) = \theta_m(\eta). \tag{10}$$

This equation is solved subject to the boundary conditions (4b). The solutions $w^*(\eta)$ and $\theta^*(\eta)$ constructed in this manner are introduced into equation (5a) for $P = 0$ and the orthogonality condition for $\theta_m(\eta)$ given by equation (8d) is utilized. One finds

$$\sum_{m=1}^{\infty} A_m\left\{\frac{\gamma_m N_m}{a^2 Ra\cos\delta}\delta_{mn} - [m/n]\right\} = 0, \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{11a}$$

where δ_{mn} is the Kronecker delta and

$$\gamma_m = a^2 + \beta_m^2,\quad [m/n] \equiv \int_0^1 w_m(\eta)\theta_n(\eta)\,d\eta. \tag{11b}$$

Equation (11a) represents a linear homogeneous system which has a solution if and only if the determinant of the coefficients A_m vanishes. This requirement leads to the following infinite order secular determinant for the evaluation of the Rayleigh number, Ra,

$$\left\|\frac{\gamma_m N_m}{a^2 Ra\cos\delta}\delta_{mn} - [m/n]\right\| = 0,\quad n = 1, 2, 3, \ldots \tag{12}$$

where the integral term, $[m/n]$, defined by equation (11b) is integrated exactly. For a given value of the system parameters "$Ra\cos\delta$" is calculated from equation (12) for several different values of the wave number "a" and the minimum value of the Rayleigh number, Ra_c, is established.

RESULTS

Figure 1 shows a plot of the critical Rayleigh number Ra_c based on the difference between the wall temperatures, ($\bar{T}_{w1}$

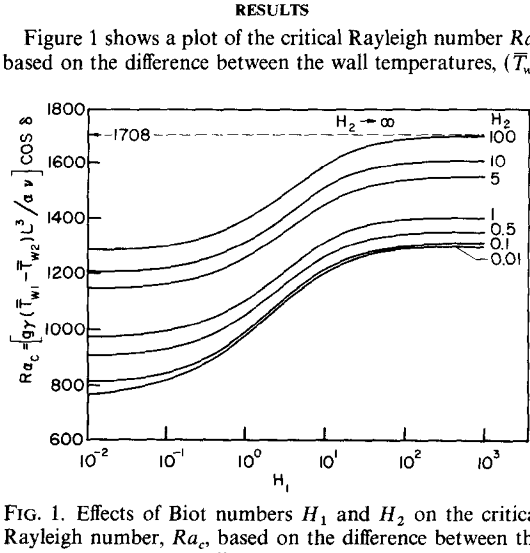

FIG. 1. Effects of Biot numbers H_1 and H_2 on the critical Rayleigh number, Ra_c, based on the difference between the wall temperatures.

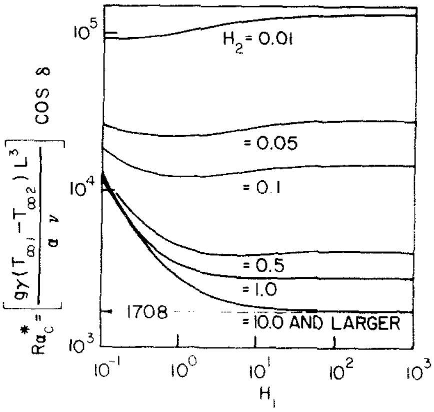

FIG. 2. Effects of Biot numbers H_1 and H_2 on the critical Rayleigh number, Ra_c^*, based on the difference between the environment temperatures.

$-\bar{T}_{w2}$). The limiting case for H_1, $H_2 \to \infty$ corresponds to the fixed temperature at the boundaries and the critical Rayleigh member $Ra_c = 1708$ is the same as that for the classical Bénard problem [1]. Ra_c decreases monotonically with decreasing Biot numbers H_1 and H_2; the most stable situation corresponds to fixed surface temperature when the Rayleigh number is based on the wall temperature difference, $(\bar{T}_{w1}-\bar{T}_{w2})$. This implies that fixing the surface temperature damps the perturbation of the temperature profile more than that with convective boundary condition.

Figure 2 shows a plot of the critical Rayleigh number Ra_c^* based on the difference between the environment temperatures, $(T_{\infty1}-T_{\infty2})$, which remains fixed as H_1 and H_2 are varied. In this case the minimum value of the critical Rayleigh number $Ra_c^* = 1708$ occurs for H_1, $H_2 \to \infty$ and it is the same as that for the Bénard problem. It is apparent from equation (6) that the relation between the two critical Rayleigh numbers is given as $Ra_c^* = Ra_c[1+(1/H_1)+(1/H_2)]$. Then, the results in Fig. 2 imply that the increase in $[1+(1/H_1)+(1/H_2)]$ by the decrease in Biot numbers is much faster than the decrease in Ra_c based on Fig. 1. Thus Ra_c^* increases as H_1 and H_2 decrease. The curves in Fig. 2 show a slight dip. The reason for this is as follows. For a given H_2, Ra_c always decreases with decreasing H_1 as apparent from Fig. 1, whereas the quantity $[1+(1/H_1)+(1/H_2)]$ always increases with H_1; then the resulting product gives rise to a slight dip in the curves in Fig. 2. Data in both of these figures is expected to be applicable for inclinations up to 75° from the horizontal for fluids having a Prandtl number about 0.7. For fluids with higher Prandtl number the transition from the longitudinal to transverse disturbances occur at angles greater than 75°. The horizontal case is applicable for all Prandtl numbers.

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